Lecture 17: Analytical Modeling of Parallel Programs: Scalability

CSCE 569 Parallel Computing

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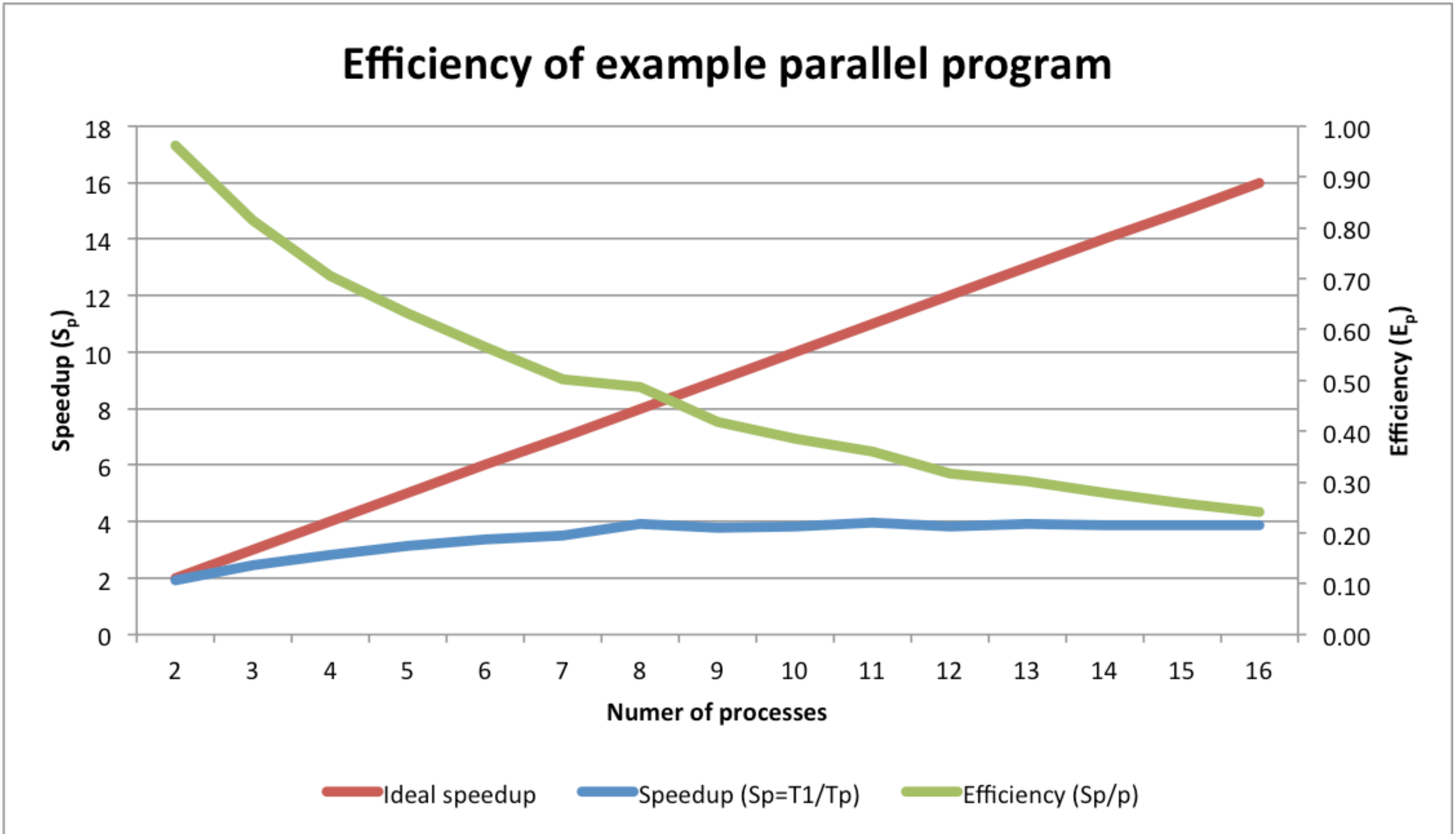
Topic Overview

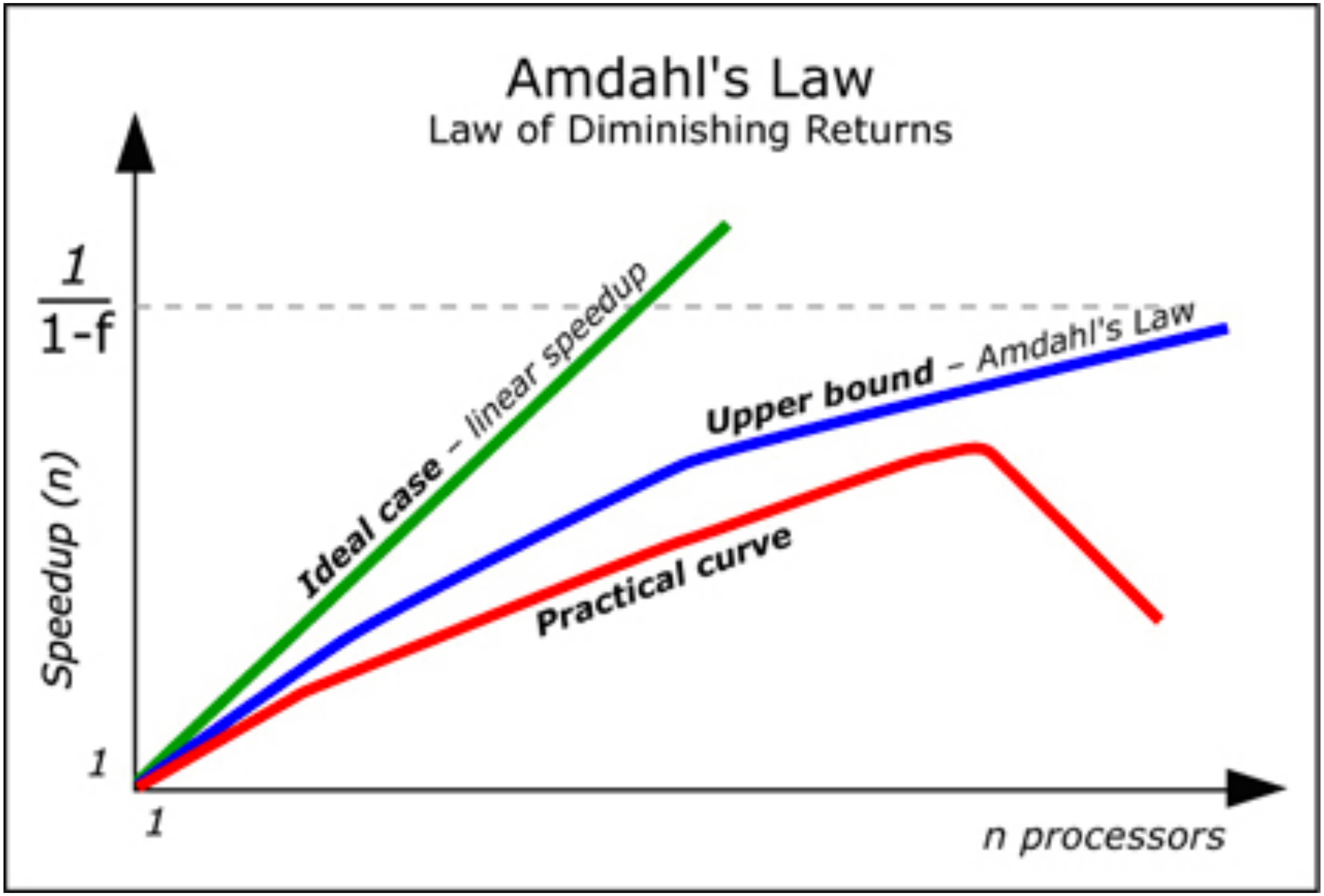
- Introduction
- Performance Metrics for Parallel Systems
  - Execution Time, Overhead, Speedup, Efficiency, Cost
- Amdahl's Law
- ☛ **Scalability of Parallel Systems**
  - **Isoefficiency Metric of Scalability**
- **Minimum Execution Time and Minimum Cost-Optimal Execution Time**
- **Asymptotic Analysis of Parallel Programs**
- **Other Scalability Metrics**
  - **Scaled speedup, Serial fraction**

Speedup and Efficiency



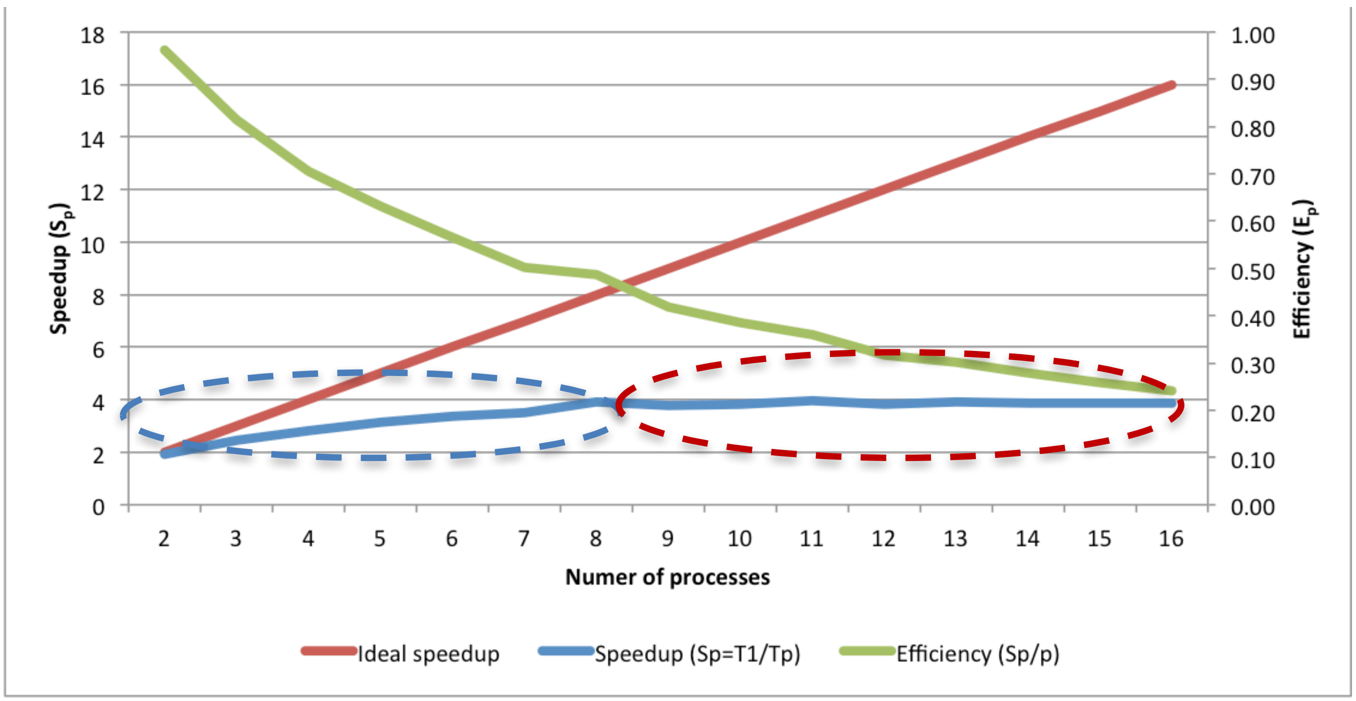
Amdahl's Law Speedup

$$S(N) \leq \frac{1}{1-F}$$



Scalability of Parallel Systems

- **Scalability: The patterns of speedup**
  - **How the performance of a parallel application changes *as the number of processors is increased***
    - ***Scaling*: performance improves *steadily***
    - ***Not scaling*: performance does not improve or becomes worse**



Scalability of Parallel Systems

- **Two different types of scaling *with regards to the problem size***
  - **Strong Scaling**
    - **Total problem size stays the same as the number of processors increases**
  - **Weak Scaling**
    - **The problem size increases at the same rate as the number of processors, keeping the amount of work per processor the same**
- **Strong scaling is generally more useful and more difficult to achieve than weak scaling**

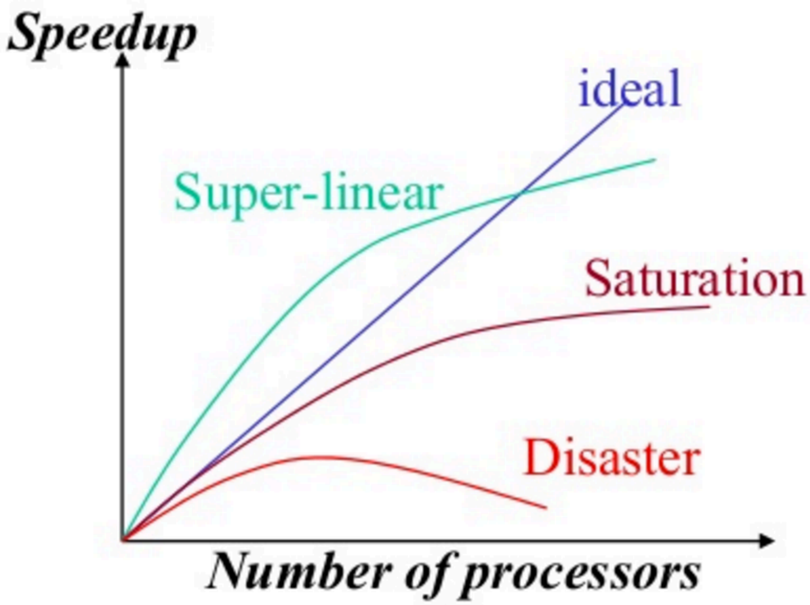
- http://www.mcs.anl.gov/~itf/dbpp/text/node30.html
- https://www.sharcnet.ca/help/index.php/Measuring_Parallel_Scaling_Performance

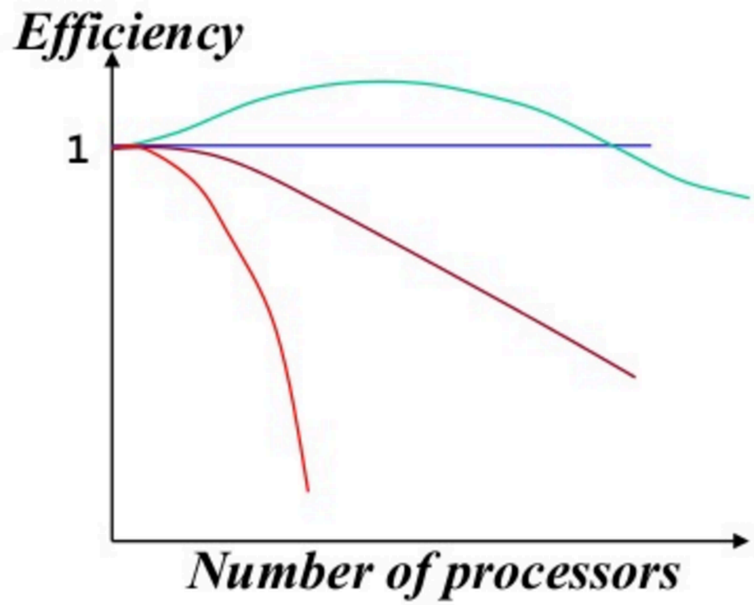
Strong Scaling

```
S(p)  =  T(1)/T(p)
E(p)  =  S(p)/p
```

for *ideal* parallel speedup we get:

```
T(p)  = T(1)/p
S(p)  = T(1)/T(p)  = p
E(p)  = S(p)/p = 1  or 100%
```



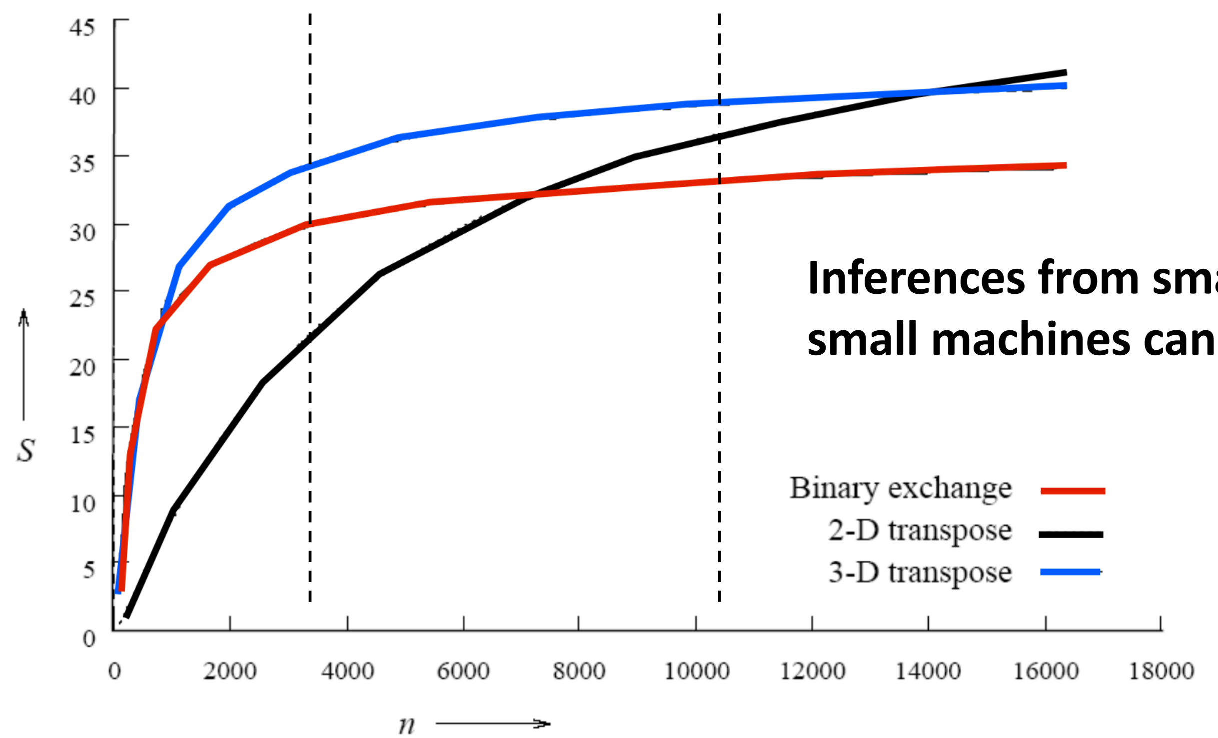


Weak Scaling of Parallel Systems

Extrapolate performance

- From small problems and small systems → larger problems on larger configurations

3 parallel algorithms for computing an n-point FFT on 64 PEs



Inferences from small datasets or small machines can be misleading

Scaling Characteristics of Parallel Programs: Increase problem size

- Efficiency: $$E = \frac{S}{p} = \frac{T_S}{pT_P}$$
- Parallel overhead: $T_o = p\,T_P - T_S \Rightarrow E = T_S / (T_S + T_o)$
  - **Overhead increases as *p* increase**

$$E = \frac{1}{1 + \frac{T_o}{T_S}}$$

- Problem size:
  - Given problem size, T_S remains constant
- **Efficiency *increases* if**
  - **The problem size increases (T_s) and**
  - **Keeping the number of PEs constant.**

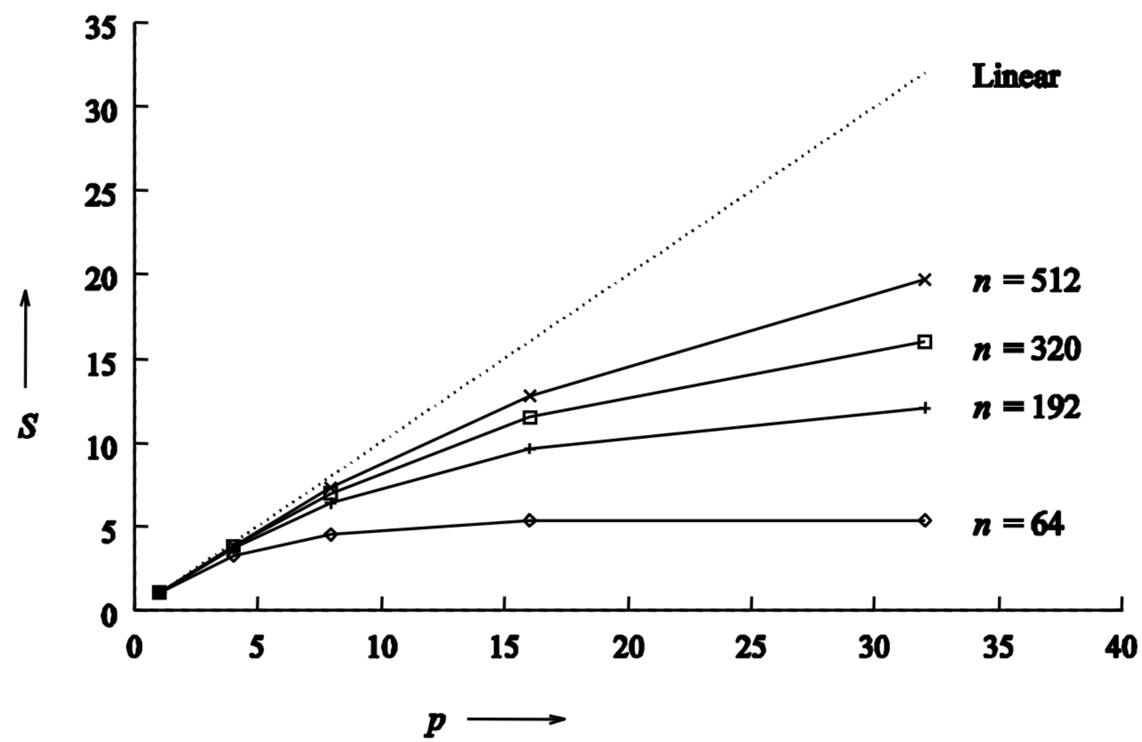
Example: Adding *n* Numbers on *p* PEs

- Addition = 1 time unit; communication = 1 time unit

$$T_P = \frac{n}{p} + 2\log p$$

$$S = \frac{n}{\frac{n}{p} + 2\log p}$$

$$E = \frac{1}{1 + \frac{2p\log p}{n}}$$



Speedup tends to saturate and efficiency drops

Scaling Characteristics of Parallel Programs: Increase problem size and increase # PEs

- Overhead $T_o = f(T_s, p)$, i.e. problem size and p
  - In many cases, T_o grows sublinearly with respect to T_s
- Efficiency:
  - Decreases as we increase p -> T_0
  - Increases as we increase problem size (Ts)

$$E = \frac{1}{1 + \frac{T_o}{T_S}}$$

- **Keep efficiency *constant***
  - **Increase problem sizes and**
  - **proportionally increasing the number of PEs**
- ***Scalable* parallel systems**

Isoefficiency Metric of Scalability

Rate at which the problem size (T_s) must increase per additional PE (T_0) to keep the efficiency fixed

$$E = \frac{1}{1 + \frac{T_o}{T_S}}$$

- The scalability of the system
  - The slower this rate, the better scalability
  - Rate == 0: strong scaling.
    - The same problem (same size) scales when increasing number of PEs

- To formalize this rate, we define
  - The problem size ***W*** = the asymptotic number of operations associated with the best serial algorithm to solve the problem.
    - The serial execution time, T_s

Isoefficiency Metric of Scalability

- Parallel overhead: $T_o(W,p)$, *again,* $W \sim= T_s$
- Parallel execution time:

$$T_P = \frac{W + T_o(W,p)}{p}$$

- Speedup:

$$S = \frac{W}{T_P} = \frac{Wp}{W + T_o(W,p)}.$$

- Efficiency

$$E = \frac{S}{p} = \frac{W}{W + T_o(W,p)} = \frac{1}{1 + T_o(W,p)/W}.$$

Isoefficiency Metric of Scalability

- **To maintain constant efficiency (between 0 and 1)**

$$E = \frac{1}{1 + T_o(W,p)/W},$$

$$\frac{T_o(W,p)}{W} = \frac{1-E}{E},$$

$$W = \frac{E}{1-E} T_o(W,p).$$

- ***K*** = ***E*** / (1 – ***E***) is a constant related to the desired efficiency

$$W = K T_o(W,p).$$

Ratio T_o / *W* should be maintained at a constant value.

Isoefficiency Metric of Scalability

$$W = KT_o(W, p).$$

$W = \Phi(p)$ such that efficiency is constant

- **$W = \Phi(p)$** is called the ***isoefficiency function***
  - Read as: what is the problem size when we have ***p*** PEs to maintain constant efficiency?
  - **$W_{p+1} - W_p = \Phi(p+1) - \Phi(p)$**
    - **To maintain constant efficiency, how much to increase the problem size if adding one more PE?**

- ***isoefficiency function*** determines the ease
  - With which a parallel system maintain a constant efficiency
  - Hence achieve speedups increasing in proportion to # PEs

Isoefficiency Example 1

Adding *n* numbers using *p* PEs

- Parallel overhead: $T_o = 2p \log p$
- $W = KT_0(W,p)$, substitute T_0
  - **W = K *2*p*log p**
- ***K *2*p*log p is* the isoefficiency function**

$$T_P = \frac{n}{p} + 2\log p$$

$$S = \frac{n}{\frac{n}{p} + 2\log p}$$

$$E = \frac{1}{1 + \frac{2p\log p}{n}}$$

- The asymptotic isoefficiency function for this parallel system is ***Θ(p*log p)***
- **To have the same efficiency on p' processors as on p**
  - **problem size *n* must increase by (p' log p') / (p log p) when increasing PEs from p to p'**

Examples

- by (p' log p') / (p log p)

- If p = 8, p' = 16
- 16*log16/(8*log8) = 16*4/(8*3) = 8/3 = 2.67

- 10M on 8 cores
- 10*2.67M on 16 cores

Cost-Optimality and Isoefficiency

- A parallel system is cost-optimal if and only if
  - Parallel cost == total work
    - Efficiency = 1

$$pT_P = \Theta(W).$$

- From this, we have:
  - i.e. work dominates overhead

$$\begin{aligned} W + T_o(W,p) &= \Theta(W) \\ T_o(W,p) &= O(W) \\ \boxed{W = \Omega(T_o(W,p))} \end{aligned}$$

- If we have an isoefficiency function ***f***(***p***)
  - The relation ***W*** = Ω(***f***(***p***)) must be satisfied to ensure the cost-optimality of a parallel system as it is scaled up

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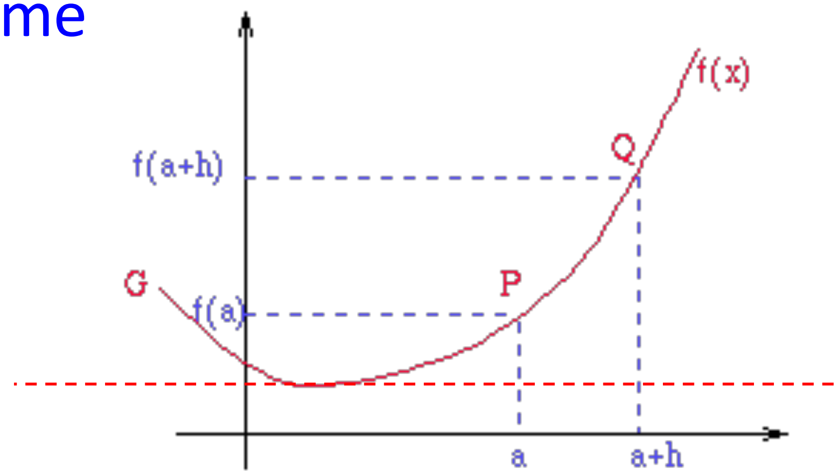
Minimum Execution Time

- Often, we are interested in the minimum time to solution
- To determine the minimum exe time T_P^{min} for a given W
  - Differentiating the expression for T_P w.r.t. p and equate it to 0

$$\frac{d}{dp} T_P = 0$$

- If p_0 is the value of p as determined by this equation
  - $T_P(p_0)$ is the minimum parallel time



Minimum Execution Time: Example

Adding n numbers

- Parallel execution time: $$T_P = \frac{n}{p} + 2\log p.$$
- Compute the derivative: $$\frac{\partial}{\partial p}\left(\frac{n}{p} + 2\log p\right) = -\frac{n}{p^2} + 2\left(\frac{1}{p}\right)$$
- Set the derivative = 0, solve for p: $$-\frac{n}{p^2} + 2\left(\frac{1}{p}\right) = 0$$
- The corresponding exe time:

$$-\frac{n}{p} + 2 = 0$$

$$p = n/2$$

$$T_P^{min} = 2\log n.$$

Note that at this point, the formulation is not cost-optimal.

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Asymptotic Analysis of Parallel Programs

Sorting a list of *n* numbers.

- The fastest serial programs: $\Theta(n \log n)$.
- Four parallel algorithms, A1, A2, A3, and A4

Algorithm	A1	A2	A3	A4
p	n^2	$\log n$	n	$\sqrt{n}$
T_P	1	n	$\sqrt{n}$	$\sqrt{n}\log n$
S	$n \log n$	$\log n$	$\sqrt{n}\log n$	$\sqrt{n}$
E	$\frac{\log n}{n}$	1	$\frac{\log n}{\sqrt{n}}$	1
pT_P	n^2	$n \log n$	$n^{1.5}$	$n \log n$

Asymptotic Analysis of Parallel Programs

Algorithm	A1	A2	A3	A4
p	n^2	$\log n$	n	$\sqrt{n}$
T_P	1	n	$\sqrt{n}$	$\sqrt{n}\log n$
S	$n\log n$	$\log n$	$\sqrt{n}\log n$	$\sqrt{n}$
E	$\frac{\log n}{n}$	1	$\frac{\log n}{\sqrt{n}}$	1
pT_P	n^2	$n\log n$	$n^{1.5}$	$n\log n$

- If metric is speed (***T_P***), algorithm A1 is the best, followed by A3, A4, and A2
- In terms of efficiency (***E***), A2 and A4 are the best, followed by A3 and A1.
- In terms of cost(***pT_p***), algorithms A2 and A4 are cost optimal, A1 and A3 are not.

- It is important to identify the analysis objectives and to use appropriate metrics!

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Scaled Speedup: Example

***n x n* matrix multiplication**

- The serial execution time: $t_c n^3$.
- The parallel execution time: $$T_P = t_c \frac{n^3}{p} + t_s \log p + 2t_w \frac{n^2}{\sqrt{p}}$$
- Speedup: $$S = \frac{t_c n^3}{t_c \frac{n^3}{p} + t_s \log p + 2t_w \frac{n^2}{\sqrt{p}}}$$

Scaled Speedup: Example (continued)

Consider memory-constrained scaled speedup.

- We have memory complexity m= $\Theta(n^2) = \Theta(p)$, or $n^2 = c \times p$.
- At this growth rate, scaled speedup ***S'*** is given by:

$$S' = \frac{t_c(c \times p)^{1.5}}{t_c\frac{(c\times p)^{1.5}}{p} + t_s \log p + 2t_w\frac{c\times p}{\sqrt{p}}} = O(p)$$

- Note that this is scalable.

Scaled Speedup: Example (continued)

Consider time-constrained scaled speedup.

- We have $\boldsymbol{T_P} = O(1) = O(\boldsymbol{n}^3 / \boldsymbol{p})$, or $\boldsymbol{n}^3 = \boldsymbol{c} \times \boldsymbol{p}$.
- Time-constrained speedup $\boldsymbol{S''}$ is given by:

$$S'' = \frac{t_c c \times p}{t_c \frac{c \times p}{p} + t_s \log p + 2 t_w \frac{(c \times p)^{2/3}}{\sqrt{p}}} = O(p^{5/6})$$

- Memory constrained scaling yields better performance.

References

- Adapted from slides “Principles of Parallel Algorithm Design” by Ananth Grama
- “Analytical Modeling of Parallel Systems”, Chapter 5 in Ananth Grama, Anshul Gupta, George Karypis, and Vipin Kumar, Introduction to Parallel Computing'', “ Addison Wesley, 2003.
- Grama, Ananth Y.; Gupta, A.; Kumar, V., "Isoefficiency: measuring the scalability of parallel algorithms and architectures," in Parallel & Distributed Technology: Systems & Applications, IEEE , vol.1, no.3, pp.12-21, Aug. 1993, doi: 10.1109/88.242438, http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=242438&isnumber=6234